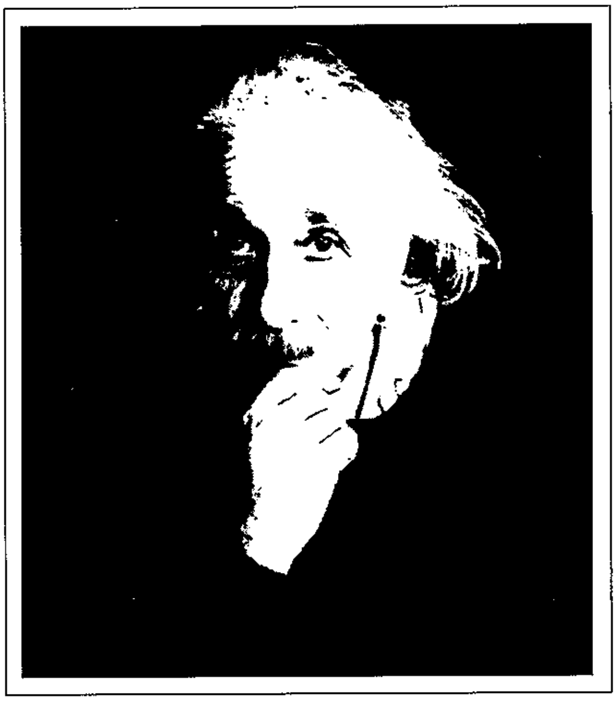
9 MATRICES AND TRANSFORMATIONS



Albert Einstein (1879–1955) working as a patent office clerk in Berne, was responsible for the greatest advance in mathematical physics of this century. His theories of relativity, put forward in 1905 and 1915 were based on the postulate that the velocity of light is absolute: mass, length and even time can only be measured relative to the observer and undergo transformation when studied by another observer. His formula $E = mc^2$ laid the foundations of nuclear physics, a fact that he came to deplore in its application to warfare. In 1933 he moved from Nazi Germany and settled in America.

36 Display information in the form of a matrix; calculate the sum and product of two matrices; calculate the product of a matrix and a scalar quantity; calculate the determinant and inverse

37 Use the following transformations: reflection, rotation, translation, enlargement, shear, stretching and their combinations; identify and give descriptions of transformations connecting given figures; describe transformations using coordinates and matrices

9.1 Matrix operations

Addition and subtraction

Matrices of the same order are added (or subtracted) by adding (or subtracting) the corresponding elements in each matrix.

Example

$$\begin{pmatrix} 2 & -4 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 5 \\ -1 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & 7 \end{pmatrix}$$

Multiplication by a number

Each element of the matrix is multiplied by the multiplying number.

Example

$$3 \times \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 3 & 12 \end{pmatrix}$$

Multiplication by another matrix

For 2×2 matrices,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix}$$

The same process is used for matrices of other orders.

Example

Perform the following multiplications.

(a) $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 6+2 & 3+10 \\ 8+1 & 4+5 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ 9 & 9 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 2+1-8 & 0-2-6 \\ 0+1+12 & 0-2+9 \end{pmatrix}$

$$= \begin{pmatrix} -5 & -8 \\ 13 & 7 \end{pmatrix}$$

Matrices may be multiplied only if they are *compatible*. The number of *columns* in the left-hand matrix must equal the number of *rows* in the right-hand matrix.

Matrix multiplication is not commutative, i.e. for square matrices **A** and **B**, the product **AB** does not necessarily equal the product **BA**.

Note
In matrices, A^2 means $A \times A$. You must multiply the matrices together.

Exercise 1

In questions **1** to **36**, the matrices have the following values:

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$; $\mathbf{B} = \begin{pmatrix} 0 & 5 \\ 1 & -2 \end{pmatrix}$; $\mathbf{C} = \begin{pmatrix} 4 & 3 \\ 1 & -2 \end{pmatrix}$; $\mathbf{D} = \begin{pmatrix} 1 & 5 & 1 \\ 4 & -6 & 1 \end{pmatrix}$; $\mathbf{E} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 5 \end{pmatrix}$;

$\mathbf{F} = (4 \quad 5)$; $\mathbf{G} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$; $\mathbf{H} = \begin{pmatrix} 0 & 1 & -2 \\ 3 & -4 & 5 \end{pmatrix}$; $\mathbf{J} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$; $\mathbf{K} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ -7 & 0 \end{pmatrix}$

Calculate the resultant value for each question where possible.

1. $\mathbf{A}+\mathbf{B}$
2. $\mathbf{D}+\mathbf{H}$
3. $\mathbf{J}+\mathbf{F}$
4. $\mathbf{B}-\mathbf{C}$
5. $2\mathbf{F}$
6. $3\mathbf{B}$

7. $\mathbf{K}-\mathbf{E}$	**8.** $2\mathbf{A}+\mathbf{B}$	**9.** $\mathbf{G}-\mathbf{J}$
10. $\mathbf{C}+\mathbf{B}+\mathbf{A}$	**11.** $2\mathbf{E}-3\mathbf{K}$	**12.** $\frac{1}{2}\mathbf{A}-\mathbf{B}$
13. $\mathbf{AB}$	**14.** $\mathbf{BA}$	**15.** $\mathbf{BC}$
16. $\mathbf{CB}$	**17.** $\mathbf{DG}$	**18.** $\mathbf{AJ}$
19. $\mathbf{HK}$	**20.** $(\mathbf{AB})\mathbf{C}$	**21.** $\mathbf{A}(\mathbf{BC})$
22. $\mathbf{AF}$	**23.** $\mathbf{CK}$	**24.** $\mathbf{GF}$
25. $\mathbf{B}(2\mathbf{A})$	**26.** $(\mathbf{D}+\mathbf{H})\mathbf{G}$	**27.** $\mathbf{JF}$
28. $\mathbf{FJ}$	**29.** $(\mathbf{A}-\mathbf{C})\mathbf{D}$	**30.** $\mathbf{A}^2$
31. $\mathbf{A}^4$	**32.** $\mathbf{E}^2$	**33.** $\mathbf{KH}$
34. $(\mathbf{CA})\,\mathbf{J}$	**35.** $\mathbf{ED}$	**36.** $\mathbf{B}^4$

In questions **37** to **46**, find the value of the letters.

37. $\begin{pmatrix} 2 & x \\ y & 7 \end{pmatrix} + \begin{pmatrix} 4 & y \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} x & 9 \\ z & 9 \end{pmatrix}$

38. $\begin{pmatrix} x & 2 \\ -1 & -2 \\ w & 3 \end{pmatrix} + \begin{pmatrix} x & y \\ y & -3 \\ v & 5 \end{pmatrix} = \begin{pmatrix} 8 & z \\ x & w \\ w & 8 \end{pmatrix}$

39. $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} - \begin{pmatrix} 2 & 5 \\ -3 & d \end{pmatrix} = 2\begin{pmatrix} 1 & a \\ b & -1 \end{pmatrix}$

40. $\begin{pmatrix} x & 3 \\ -2 & y \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

41. $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$

42. $\begin{pmatrix} p & 2 & -1 \\ q & -2 & 2q \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \end{pmatrix}$

43. $\begin{pmatrix} 3 & 0 \\ 2 & x \end{pmatrix}\begin{pmatrix} y & z \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 8 & w \end{pmatrix}$

44. $\begin{pmatrix} 3y & 3z \\ 2y+4x & 2z \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 8 & w \end{pmatrix}$

45. $\begin{pmatrix} 2 & e \\ a & 3 \end{pmatrix} + k\begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ -3 & -1 \end{pmatrix}$

46. $\begin{pmatrix} 4 & 0 \\ 1 & m \end{pmatrix}\begin{pmatrix} n & p \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 20 & 12 \\ -1 & q \end{pmatrix}$

47. If $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} x & 0 \\ 1 & 3 \end{pmatrix}$, and $\mathbf{AB} = \mathbf{BA}$, find x.

48. If $\mathbf{X} = \begin{pmatrix} k & 2 \\ 2 & -k \end{pmatrix}$ and $\mathbf{X}^2 = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find k.

49. $\mathbf{B} = \begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$

(a) Find k if $\mathbf{B}^2 = k\mathbf{B}$
(b) Find m if $\mathbf{B}^4 = m\mathbf{B}$

50. $\mathbf{A} = \begin{pmatrix} 5 & 5 \\ -2 & -2 \end{pmatrix}$

(a) Find n if $\mathbf{A}^2 = n\mathbf{A}$
(b) Find q if $\mathbf{A}^3 = q\mathbf{A}$

9.2 The inverse of a matrix

The inverse of a matrix **A** is written $\mathbf{A}^{-1}$, and the inverse exists if

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

where **I** is called the identity matrix.

Only square matrices possess an inverse.

For 2×2 matrices, $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

For 3×3 matrices, $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, etc.

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse $\mathbf{A}^{-1}$ is given by $\mathbf{A}^{-1} = \dfrac{1}{(ad-cb)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Here, the number $(ad - cb)$ is called the *determinant* of the matrix and is written $|\mathbf{A}|$.

If $|\mathbf{A}| = 0$, then the matrix has no inverse.

Example

Find the inverse of $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix}$.

$$\mathbf{A}^{-1} = \frac{1}{[3(-2) - 1(-4)]}\begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$$

Check: $\mathbf{A}^{-1}\mathbf{A} = \dfrac{1}{-2}\begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix} = \dfrac{-1}{2}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiplying by the inverse of a matrix gives the same result as dividing by the matrix: the effect is similar to ordinary algebraic operations.

e.g. if
$$\begin{aligned} \mathbf{AB} &= \mathbf{C} \\ \mathbf{A}^{-1}\mathbf{AB} &= \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{B} &= \mathbf{A}^{-1}\mathbf{C} \end{aligned}$$

Exercise 2

In questions **1** to **15**, find the inverse of the matrix.

1. $\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$ **2.** $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ **3.** $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

4. $\begin{pmatrix} 5 & 2 \\ 1 & 1 \end{pmatrix}$ **5.** $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ **6.** $\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$

7. $\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$ **8.** $\begin{pmatrix} 0 & -3 \\ 2 & 4 \end{pmatrix}$ **9.** $\begin{pmatrix} -1 & -2 \\ 1 & -3 \end{pmatrix}$

10. $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ **11.** $\begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}$ **12.** $\begin{pmatrix} -3 & 1 \\ 2 & 1 \end{pmatrix}$

13. $\begin{pmatrix} 2 & -3 \\ 1 & -4 \end{pmatrix}$ **14.** $\begin{pmatrix} 7 & 0 \\ -5 & 1 \end{pmatrix}$ **15.** $\begin{pmatrix} 2 & 1 \\ -2 & -4 \end{pmatrix}$

16. If $\mathbf{B} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{AB} = \mathbf{I}$, find $\mathbf{A}$.

17. Find $\mathbf{Y}$ if $\mathbf{Y}\begin{pmatrix} -2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

18. If $\begin{pmatrix} 2 & -3 \\ 0 & 4 \end{pmatrix} + \mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find $\mathbf{X}$.

19. Find $\mathbf{B}$ if $\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 4 & -2 \\ 0 & 7 \end{pmatrix}$.

20. If $\begin{pmatrix} 3 & -3 \\ 2 & 5 \end{pmatrix} - \mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find $\mathbf{X}$.

21. Find $\mathbf{M}$ if $\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\mathbf{M} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

22. $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$.

Find: (a) $\mathbf{AB}$ (b) $\mathbf{A}^{-1}$ (c) $\mathbf{B}^{-1}$

Show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

23. If $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$ and $\mathbf{MN} = \begin{pmatrix} 7 & -9 \\ -2 & -6 \end{pmatrix}$, find $\mathbf{N}$.

24. $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$; $\mathbf{C} = \begin{pmatrix} 11 \\ 7 \end{pmatrix}$. If $\mathbf{B}$ is a (2×1) matrix such that $\mathbf{AB} = \mathbf{C}$, find $\mathbf{B}$.

25. Find x if the determinant of $\begin{pmatrix} x & 3 \\ 1 & 2 \end{pmatrix}$ is

(a) 5 (b) -1 (c) 0

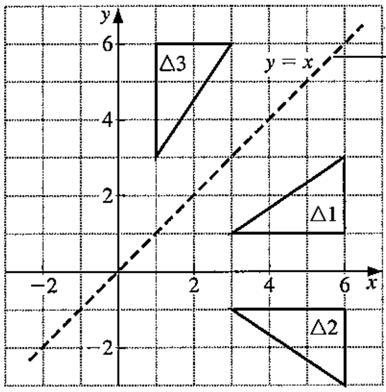
26. If the matrix $\begin{pmatrix} 1 & -2 \\ x & 4 \end{pmatrix}$ has no inverse, what is the value of x?

27. The elements of a (2×2) matrix consist of four different numbers. Find the largest possible value of the determinant of this matrix if the numbers are:

(a) 1, 3, 5, 9 (b) -1, 2, 3, 4

9.3 Simple transformations

Reflection



Remember:
You must describe a mirror line fully.

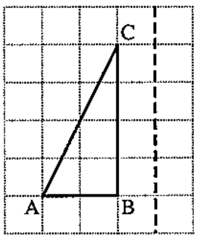
Δ2 is the image of Δ1 after reflection in the x-axis.

Δ3 is the image of Δ1 after reflection in the line $y = x$.

Exercise 3

In questions **1** to **6** draw the object and its image after reflection in the broken line.

1.



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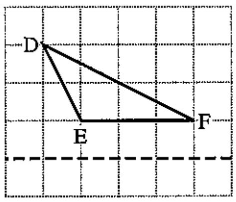

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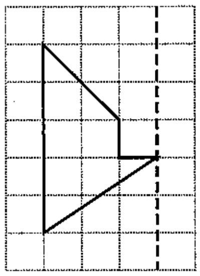

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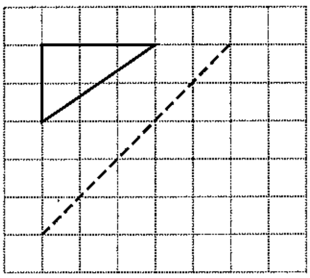

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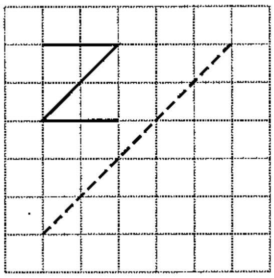

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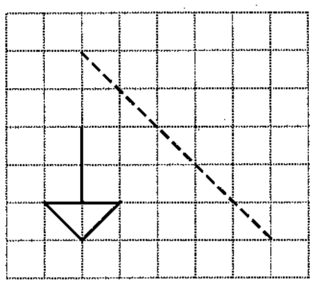
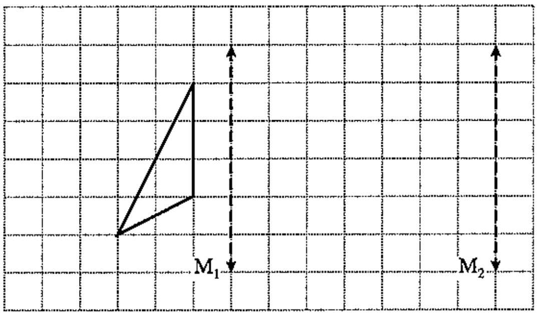


In questions **7**, **8**, **9** draw the image of the given shape after reflection in line M_1 and then reflect this new shape in line M_2.

7.



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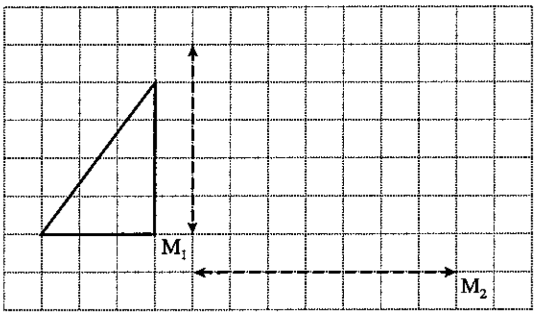

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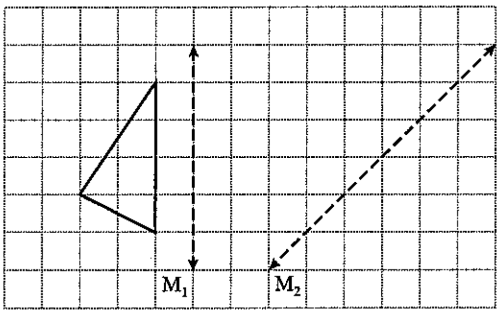


Exercise 4

For each question draw x- and y-axes with values from -8 to 8.

1. (a) Draw the triangle ABC at A(6, 8), B(2, 8), C(2, 6). Draw the lines $y = 2$ and $y = x$.

(b) Draw the image of $\triangle$ABC after reflection in:
 (i) the y-axis. Label it $\triangle 1$.
 (ii) the line $y = 2$. Label it $\triangle 2$.
 (iii) the line $y = x$. Label it $\triangle 3$.

(c) Write down the coordinates of the image of point A in each case.

2. (a) Draw the triangle DEF at D(−6, 8), E(−2, 8), F(−2, 6). Draw the lines $x = 1$, $y = x$, $y = -x$.

(b) Draw the image of $\triangle$DEF after reflection in:
(i) the line $x = 1$. Label it $\triangle 1$.
(ii) the line $y = x$. Label it $\triangle 2$.
(iii) the line $y = -x$. Label it $\triangle 3$.

(c) Write down the coordinates of the image of point D in each case.

3. (a) Draw the triangle ABC at A(5, 1), B(8, 1), C(8, 3). Draw the lines $x + y = 4$, $y = x - 3$, $x = 2$.

(b) Draw the image of $\triangle$ABC after reflection in:
(i) the line $x + y = 4$. Label it $\triangle 1$.
(ii) the line $y = x - 3$. Label it $\triangle 2$.
(iii) the line $x = 2$. Label it $\triangle 3$.

(c) Write down the coordinates of the image of point A in each case.

4. (a) Draw and label the following triangles:
(i) $\triangle 1 : (3, 3), (3, 6), (1, 6)$
$\triangle 2 : (3, -1), (3, -4), (1, -4)$
$\triangle 3 : (3, 3), (6, 3), (6, 1)$
$\triangle 4 : (-6, -1), (-6, -3), (-3, -3)$
$\triangle 5 : (-6, 5), (-6, 7), (-3, 7)$

(b) Find the equation of the mirror line for the reflection:
(i) $\triangle 1$ onto $\triangle 2$ (ii) $\triangle 1$ onto $\triangle 3$
(iii) $\triangle 1$ onto $\triangle 4$ (iv) $\triangle 4$ onto $\triangle 5$

5. (a) Draw $\triangle 1$ at (3, 1), (7, 1), (7, 3).

(b) Reflect $\triangle 1$ in the line $y = x$ onto $\triangle$ 2.

(c) Reflect $\triangle 2$ in the x-axis onto $\triangle 3$.

(d) Reflect $\triangle 3$ in the line $y = -x$ onto $\triangle 4$.

(e) Reflect $\triangle 4$ in the line $x = 2$ onto $\triangle 5$.

(f) Write down the coordinates of $\triangle 5$.

6. (a) Draw $\triangle 1$ at (2, 6), (2, 8), (6, 6).

(b) Reflect $\triangle 1$ in the line $x + y = 6$ onto $\triangle 2$.

(c) Reflect $\triangle 2$ in the line $x = 3$ onto $\triangle 3$.

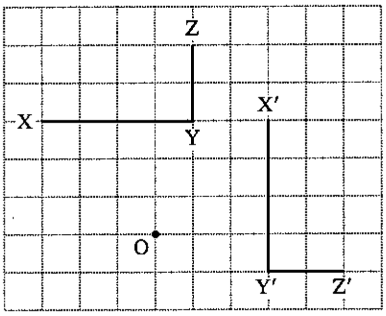
(d) Reflect $\triangle 3$ in the line $x + y = 6$ onto $\triangle 4$.

(e) Reflect $\triangle 4$ in the line $y = x - 8$ onto $\triangle 5$.

(f) Write down the coordinates of $\triangle 5$.

Rotation

Example



The letter L has been rotated through 90° clockwise about the centre O. The angle, direction, and centre are needed to fully describe a rotation.

We say that the object *maps* onto the image. Here,

X maps onto X′
Y maps onto Y′
Z maps onto Z′

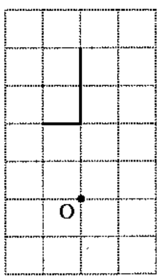
In this work, a clockwise rotation is *negative* and an anticlockwise rotation is *positive*: in this example, the letter L has been rotated through −90°. The angle, the direction, and the centre of rotation can be found using tracing paper and a sharp pencil placed where you think the centre of rotation is.

For more accurate work, draw the perpendicular bisector of the line joining two corresponding points, e.g. Y and Y′. Repeat for another pair of corresponding points. The centre of rotation is at the intersection of the two perpendicular bisectors.

Exercise 5

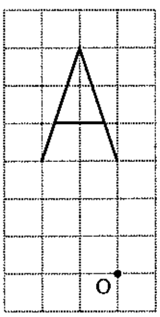
In questions **1** to **4** draw the object and its image under the rotation given. Take O as the centre of rotation in each case.

1.



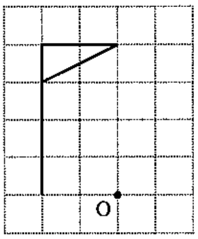
90° clockwise

2.



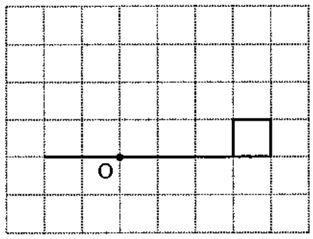
90° anticlockwise

3.



180°

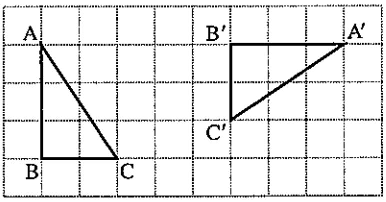
4.



90° clockwise

In questions **5** to **8**, copy the diagram on squared paper and find the angle, the direction, and the centre of the rotation.

5.



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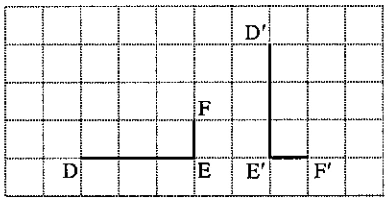

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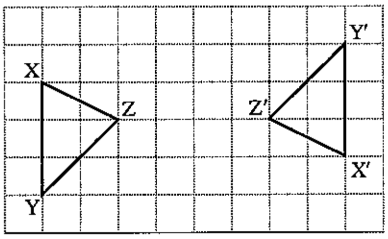

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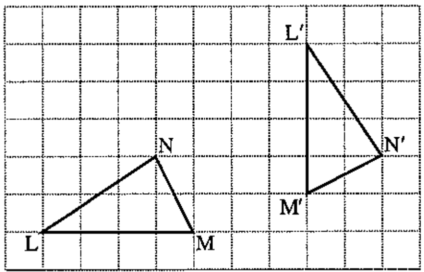


Exercise 6

For all questions draw x- and y-axes for values from -8 to $+8$.

1. (a) Draw the object triangle ABC at A(1, 3), B(1, 6), C(3, 6), rotate ABC through 90° clockwise about (0, 0), mark A′B′C′.
(b) Draw the object triangle DEF at D(3, 3), E(6, 3), F(6, 1), rotate DEF through 90° clockwise about (0, 0), mark D′E′F′.
(c) Draw the object triangle PQR at P(−4, 7), Q(−4, 5), R(−1, 5), rotate PQR through 90° anticlockwise about (0, 0), mark P′Q′R′.

2. (a) Draw $\triangle 1$ at (1, 4), (1, 7), (3, 7).
(b) Draw the images of $\triangle 1$ under the following rotations:
(i) 90° clockwise, centre (0, 0). Label it $\triangle 2$.
(ii) 180°, centre (0, 0). Label it $\triangle 3$.
(iii) 90° anticlockwise, centre (0, 0). Label it $\triangle 4$.

3. (a) Draw triangle PQR at P(1, 2), Q(3, 5), R(6, 2).
(b) Find the image of PQR under the following rotations:
(i) 90° anticlockwise, centre (0, 0); label the image P′Q′R′
(ii) 90° clockwise, centre (−2, 2); label the image P″Q″R″
(iii) 180°, centre (1, 0); label the image P*Q*R*.
(c) Write down the coordinates of P′, P″, P*.

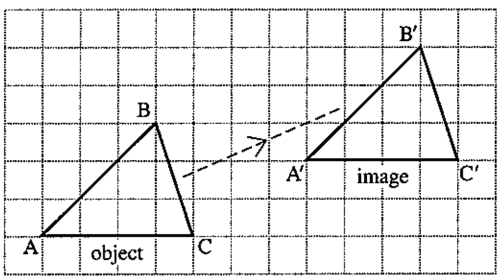
4. (a) Draw $\triangle 1$ at (1, 2), (1, 6), (3, 5).
(b) Rotate $\triangle 1$ 90° clockwise, centre (1, 2) onto $\triangle 2$.
(c) Rotate $\triangle 2$ 180°, centre (2, −1) onto $\triangle 3$.
(d) Rotate $\triangle 3$ 90° clockwise, centre (2, 3) onto $\triangle 4$.
(e) Write down the coordinates of $\triangle 4$.

5. (a) Draw and label the following triangles:
$\triangle 1$: (3, 1), (6, 1), (6, 3)
$\triangle 2$: (−1, 3), (−1, 6), (−3, 6)
$\triangle 3$: (1, 1), (−2, 1), (−2, −1)
$\triangle 4$: (3, −1), (3, −4), (5, −4)
$\triangle 5$: (4, 4), (1, 4), (1, 2)
(b) Describe fully the following rotations:
(i) $\triangle 1$ onto $\triangle 2$ (ii) $\triangle 1$ onto $\triangle 3$
(iii) $\triangle 1$ onto $\triangle 4$ (iv) $\triangle 1$ onto $\triangle 5$
(v) $\triangle 5$ onto $\triangle 4$ (vi) $\triangle 3$ onto $\triangle 2$

6. (a) Draw $\triangle 1$ at (4, 7), (8, 5), (8, 7).
(b) Rotate $\triangle 1$ 90° clockwise, centre (4, 3) onto $\triangle 2$.
(c) Rotate $\triangle 2$ 180°, centre (5, −1) onto $\triangle 3$.
(d) Rotate $\triangle 3$ 90° anticlockwise, centre (0, −8) onto $\triangle 4$
(e) Describe fully the following rotations:
(i) $\triangle 4$ onto $\triangle 1$
(ii) $\triangle 4$ onto $\triangle 2$

Translation

The triangle ABC below has been transformed onto the triangle A′B′C′ by a *translation*.



Here the translation is 7 squares to the right and 2 squares up the page. The translation can be described by a column vector.

In this case the translation is $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$.

Exercise 7

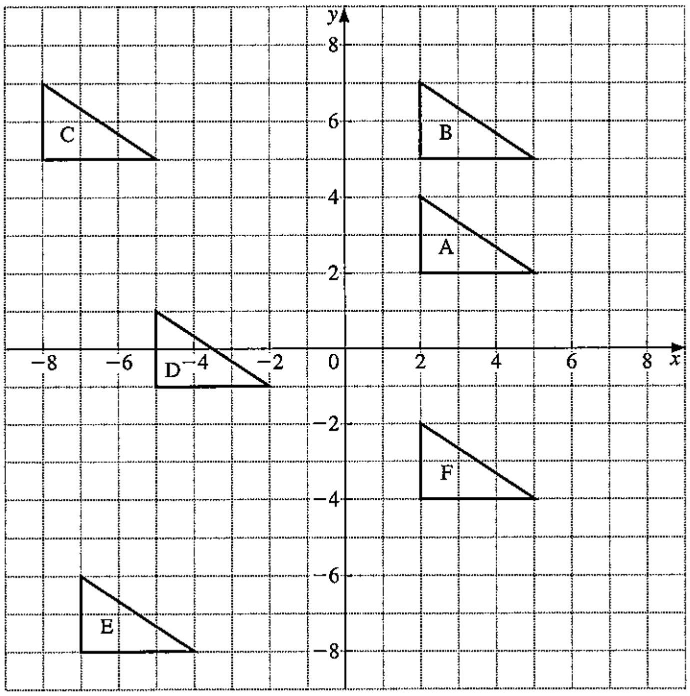
1. Make a copy of the diagram below and write down the column vector for each of the following translations:

(a) D onto A (b) B onto F

(c) E onto A (d) A onto C

(e) E onto C (f) C onto B

(g) F onto E (h) B onto C.



For questions **2** to **11** draw x and y axes with values from -8 to 8. Draw object triangle ABC at A(-4, -1), B(-4, 1), C(-1, -1) and shade it.

Draw the image of ABC under the translations described by the vectors below. For each question, write down the new coordinates of point C.

2. $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$ **3.** $\begin{pmatrix} 6 \\ 7 \end{pmatrix}$ **4.** $\begin{pmatrix} 9 \\ -4 \end{pmatrix}$ **5.** $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$

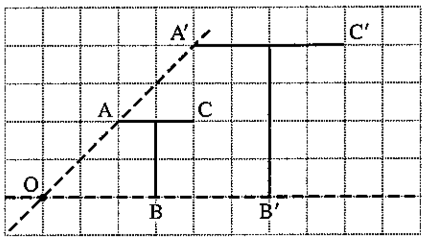
6. $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ **7.** $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ **8.** $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$ **9.** $\begin{pmatrix} 0 \\ -7 \end{pmatrix}$

10. $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ followed by $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

11. $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ followed by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ followed by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Enlargement

In the diagram below, the letter T has been enlarged by a scale factor of 2 using the point O as the centre of the enlargement.

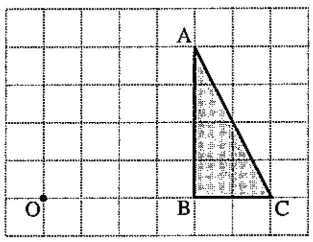


Notice that $OA' = 2 \times OA$
$OB' = 2 \times OB$

The scale factor and the centre of enlargement are both required to describe an enlargement.

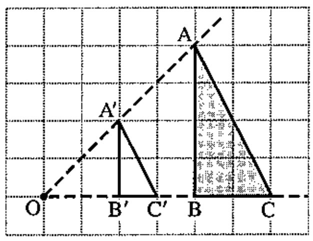
Example 1

Draw the image of triangle ABC under an enlargement scale factor of $\frac{1}{2}$ using O as centre of enlargement.

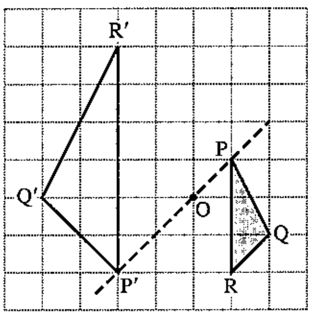


(a) Draw lines through OA, OB and OC.
(b) Mark A′ so that $OA' = \frac{1}{2}OA$
Mark B′ so that $OB' = \frac{1}{2}OB$
Mark C′ so that $OC' = \frac{1}{2}OC$.
(c) Join A′B′C as shown.

Remember always to measure the lengths from O, not from A, B, or C.



Example 2

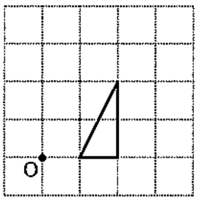


P′Q′R′ is the image of PQR after enlargement with scale factor −2 and centre O. Notice that P′ and P are on opposite sides of point O. Similarly Q′ and Q, R′ and R.

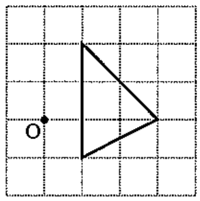
Exercise 8

In questions **1** to **6** copy the diagram and draw an enlargement using the centre O and the scale factor given.

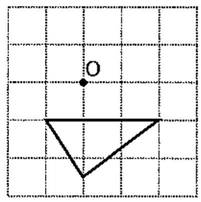
1. Scale factor 2



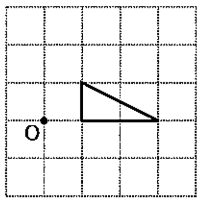
2. Scale factor 3



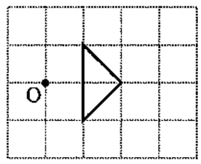
3. Scale factor 3



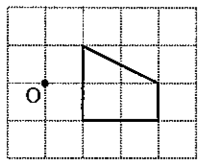
4. Scale factor −2



5. Scale factor −3



6. Scale factor $1\frac{1}{2}$



Answer questions **7** to **19** on graph paper taking x and y from 0 to 15. The vertices of the object are given in coordinate form.

In questions **7** to **10**, enlarge the object with the centre of enlargement and scale factor indicated.

	object	*centre*	*scale factor*
7.	(2, 4) (4, 2) (5, 5)	(0, 0)	+2
8.	(2, 4) (4, 2) (5, 5)	(1, 2)	+2
9.	(1, 1) (4, 2) (2, 3)	(1, 1)	+3
10.	(4, 4) (7, 6) (9, 3)	(7, 4)	+2

In questions **11** to **14** plot the object and image and find the centre of enlargement and the scale factor.

11. object A(2, 1), B(5, 1), C(3, 3)
image A′(2, 1), B′(11, 1), C′(5, 7)

12. object A(2, 5), B(9, 3), C(5, 9)
image A′($6\frac{1}{2}$, 7), B′(10, 6), C′(8, 9)

13. object A(2, 2), B(4, 4), C(2, 6)
image A′(11, 8), B′(7, 4), C′(11, 0)

14. object A(0, 6), B(4, 6), C(3, 0)
image A′(12, 6), B′(8, 6), C′(9, 12)

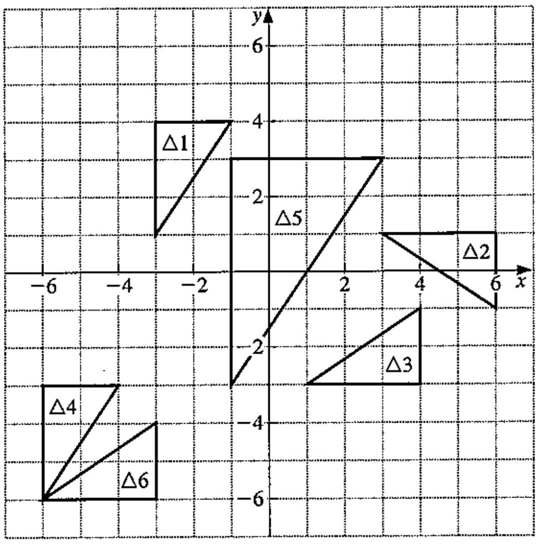
In questions **15** to **19** enlarge the object using the centre of enlargement and scale factor indicated.

	object	*centre*	*S.F.*
15.	(1, 2), (13, 2), (1, 10)	(0, 0)	$+\frac{1}{2}$
16.	(5, 10), (5, 7), (11, 7)	(2, 1)	$+\frac{1}{3}$
17.	(7, 3), (9, 3), (7, 8)	(5, 5)	−1
18.	(1, 1), (3, 1), (3, 2)	(4, 3)	−2
19.	(9, 2), (14, 2), (14, 6)	(7, 4)	$-\frac{1}{2}$

The next exercise contains questions involving the four basic transformations: reflection, rotation, translation, enlargement.

Exercise 9

1. (a) Copy the diagram below.



(b) Describe fully the following transformations:

(i) △1 → △2 (ii) △1 → △3
(iii) △4 → △1 (iv) △1 → △5
(v) △3 → △6 (vi) △6 → △4

2. Plot and label the following triangles:

△1: (−5, −5), (−1, −5), (−1, −3) △2: (1, 7), (1, 3), (3, 3)
△3: (3, −3), (7, −3), (7, −1) △4: (−5, −5), (−5, −1), (−3, −1)
△5: (1, −6), (3, −6), (3, −5) △6: (−3, 3), (−3, 7), (−5, 7)

Describe fully the following transformations:

(a) △1 → △2 (b) △1 → △3
(c) △1 → △4 (d) △1 → △5
(e) △1 → △6 (f) △5 → △3
(g) △2 → △3

3. Plot and label the following triangles:

△1: (−3, −6), (−3, −2), (−5, −2) △2: (−5, −1), (−5, −7), (−8, −1)
△3: (−2, −1), (2, −1), (2, 1) △4: (6, 3), (2, 3), (2, 5)
△5: (8, 4), (8, 8), (6, 8) △6: (−3, 1), (−3, 3), (−4, 3)

Describe fully the following transformations:

(a) △1 → △2 (b) △1 → △3
(c) △1 → △4 (d) △1 → △5
(e) △1 → △6 (f) △3 → △5
(g) △6 → △2

9.4 Combined transformations

It is convenient to denote transformations by a symbol.

Let **A** denote 'reflection in line $x = 3$' and

B denote 'translation $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$'.

Perform **A** on △1.

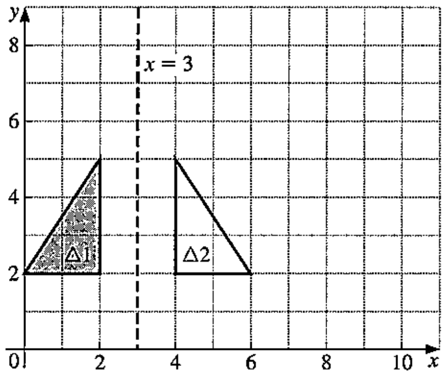


Figure 1

△2 is the image of △1 under the reflection in $x = 3$

i.e. **A**(△1) = △2

A(△1) means 'perform the transformation **A** on triangle △1'

Perform **B** on △2.

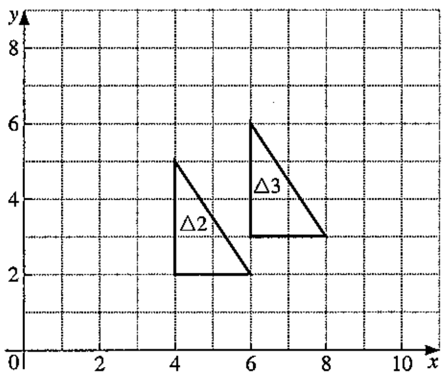


Figure 2

From Figure 2 we can see that

B(△2) = △3

The effect of going from △1 to △3 may be written

BA(△1) = △3

It is very important to notice that **BA**(△1) means do **A** first and then **B**.

Repeated transformations

XX(P) means 'perform transformation **X** on P and then perform **X** on the image'.

It may be written $\mathbf{X}^2$(P)

Similarly **TTT**(P) = $\mathbf{T}^3$(P).

Inverse transformations

If translation **T** has vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, the translation which has the opposite effect has vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$. This is written $\mathbf{T}^{-1}$.

If rotation **R** denotes 90° clockwise rotation about (0, 0), then $\mathbf{R}^{-1}$ denotes 90° *anti*clockwise rotation about (0, 0).

The *inverse* of a transformation is the transformation which takes the *image* back to the object.

Note:
For all reflections, the inverse is the same reflection.
e.g. if **X** is reflection in $x = 0$, then $\mathbf{X}^{-1}$ is also reflection in $x = 0$.

The symbol $\mathbf{T}^{-3}$ means $(\mathbf{T}^{-1})^3$ i.e. perform $\mathbf{T}^{-1}$ three times.

Exercise 10

Draw x- and y-axes with values from −8 to +8 and plot the point P(3, 2).
R denotes 90° clockwise rotation about (0, 0);
X denotes reflection in $x = 0$.
H denotes 180° rotation about (0, 0);
T denotes translation $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

For each question, write down the coordinates of the final image of P.

1. **R**(P)	**2.** **TR**(P)	**3.** **T**(P)	**4.** **RT**(P)
5. **TH**(P)	**6.** **XT**(P)	**7.** **HX**(P)	**8.** **XX**(P)
9. $\mathbf{R}^{-1}$(P)	**10.** $\mathbf{T}^{-1}$(P)	**11.** $\mathbf{X}^3$(P)	**12.** $\mathbf{T}^{-2}$(P)
13. $\mathbf{R}^2$(P)	**14.** $\mathbf{T}^{-1}\mathbf{R}^2$(P)	**15.** **THX**(P)	**16.** $\mathbf{R}^3$(P)
17. $\mathbf{TX}^{-1}$(P)	**18.** $\mathbf{T}^3\mathbf{X}$(P)	**19.** $\mathbf{T}^2\mathbf{H}^{-1}$(P)	**20.** **XTH**(P)

Exercise 11

In this exercise, transformations **A**, **B**, ... **H**, are as follows:

A denotes reflection in $x = 2$
B denotes 180° rotation, centre (1, 1)
C denotes translation $\begin{pmatrix} -6 \\ 2 \end{pmatrix}$
D denotes reflection in $y = x$
E denotes reflection in $y = 0$
F denotes translation $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$
G denotes 90° rotation clockwise, centre (0, 0)
H denotes enlargement, scale factor $+\frac{1}{2}$, centre (0, 0)

Draw x- and y-axes with values from −8 to +8.

1. Draw triangle LMN at L(2, 2), M(6, 2), N(6, 4).
   Find the image of LMN under the following combinations of transformations. Write down the coordinates of the image of point L in each case:
   (a) **CA**(LMN) (b) **ED**(LMN) (c) **DB**(LMN)
   (d) **BE**(LMN) (e) **EB**(LMN)

2. Draw triangle PQR at P(2, 2), Q(6, 2), R(6, 4).
   Find the image of PQR under the following combinations of transformations. Write down the coordinates of the image of point P in each case:
   (a) **AF**(PQR) (b) **CG**(PQR)
   (c) **AG**(PQR) (d) **HE**(PQR)

3. Draw triangle XYZ at X(−2, 4), Y(−2, 1), Z(−4, 1). Find the image of XYZ under the following combinations of transformations and state the equivalent single transformation in each case:
   (a) $\mathbf{G}^2\mathbf{E}$(XYZ) (b) **CB**(XYZ) (c) **DA**(XYZ)

4. Draw triangle OPQ at O(0, 0), P(0, 2), Q(3, 2).
   Find the image of OPQ under the following combinations of transformations and state the equivalent single transformation in each case:
   (a) **DE**(OPQ) (b) **FC**(OPQ)
   (c) **DEC**(OPQ) (d) **DFE**(OPQ)

5. Draw triangle RST at R(−4, −1), S($-2\frac{1}{2}$, −2), T(−4, −4). Find the image of RST under the following combinations of transformations and state the equivalent single transformation in each case:
   (a) **EAG**(RST) (b) **FH**(RST) (c) **GF**(RST)

6. Write down the inverses of the transformations **A**, **B**, ... **H**.

7. Draw triangle JKL at J(−2, 2), K(−2, 5), L(−4, 5). Find the image of JKL under the following transformations. Write down the coordinates of the image of point J in each case:
(a) $\mathbf{C}^{-1}$ (b) $\mathbf{F}^{-1}$ (c) $\mathbf{G}^{-1}$ (d) $\mathbf{D}^{-1}$ (e) $\mathbf{A}^{-1}$

8. Draw triangle PQR at P(−2, 4), Q(−2, 1), R(−4, 1). Find the image of PQR under the following combinations of transformations. Write down the coordinates of the image of point P in each case:
(a) $\mathbf{DF}^{-1}$(PQR) (b) $\mathbf{EC}^{-1}$(PQR) (c) $\mathbf{D}^2\mathbf{F}$(PQR)
(d) **GA**(PQR) (e) $\mathbf{C}^{-1}\mathbf{G}^{-1}$(PQR)

9. Draw triangle LMN at L(−2, 4), M(−4, 1), N(−2, 1). Find the image of LMN under the following combinations of transformations. Write down the coordinates of the image of point L in each case:
(a) **HE**(LMN) (b) $\mathbf{EAG}^{-1}$(LMN)
(c) **EDA**(LMN) (d) $\mathbf{BG}^2\mathbf{E}$(LMN)

10. Draw triangle XYZ at X(1, 2), Y(1, 6), Z(3, 6).
(a) Find the image of XYZ under each of the transformations **BC** and **CB**.
(b) Describe fully the single transformation equivalent to **BC**.
(c) Describe fully the transformation **M** such that **MCB** = **BC**.

9.5 Transformations using matrices

Example 1

Find the image of triangle ABC, with A(1, 1), B(3, 1), C(3, 2), under the transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Write the coordinates of A as a column vector and multiply this vector by **M**.

$$\underset{\mathbf{M}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \underset{\mathrm{A}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \underset{\mathrm{A}'}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

A′, the image of A, has coordinates (1, −1).

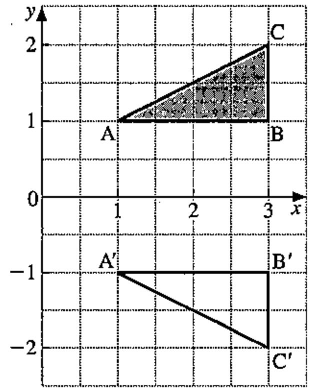
(b) Repeat for B and C.

$$\underset{\mathbf{M}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \underset{\mathrm{B}}{\begin{pmatrix} 3 \\ 1 \end{pmatrix}} = \underset{\mathrm{B}'}{\begin{pmatrix} 3 \\ -1 \end{pmatrix}}$$

$$\underset{\mathbf{M}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \underset{\mathrm{C}}{\begin{pmatrix} 3 \\ 2 \end{pmatrix}} = \underset{\mathrm{C}'}{\begin{pmatrix} 3 \\ -2 \end{pmatrix}}$$

(c) Plot A′(1, −1), B′(3, −1) and C′(3, −2).

The transformation is a reflection in the x-axis.



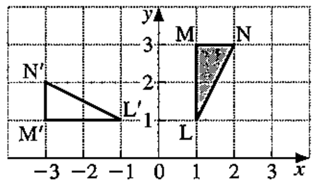
Example 2

Find the image of L(1, 1), M(1, 3), N(2, 3) under the transformation represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

A quicker method is to write the three vectors for L, M and N in a single 2 × 3 matrix, and then perform the multiplication.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} \mathrm{L} & \mathrm{M} & \mathrm{N} \end{array} \\ \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} \mathrm{L}' & \mathrm{M}' & \mathrm{N}' \end{array} \\ \begin{pmatrix} -1 & -3 & -3 \\ 1 & 1 & 2 \end{pmatrix} \end{array}$$

The transformation is a rotation, +90°, centre (0, 0).



Exercise 12

For questions **1** to **5** draw x- and y-axes with values from −8 to 8. Do all parts of each question on one graph.

1. Draw the triangle A(2, 2), B(6, 2), C(6, 4). Find its image under the transformations represented by the following matrices:

(a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

2. Plot the object and image for the following:

	Object	*Matrix*
(a)	P(4, 2), Q(4, 4), R(0, 4)	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
(b)	P(4, 2), Q(4, 4), R(0, 4)	$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$
(c)	A(−6, 8), B(−2, 8), C(−2, 6)	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
(d)	P(4, 2), Q(4, 4), R(0, 4)	$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

Describe each as a *single* transformation.

3. Draw a trapezium at K(2, 2), L(2, 5), M(5, 8), N(8, 8). Find the image of KLMN under the transformations described by the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{G} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Describe fully each of the eight transformations.

4. (a) Draw a quadrilateral at A(3, 4), B(4, 0), C(3, 1), D(0, 0). Find the image of ABCD under the transformation represented by the matrix $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$.
   (b) Find the ratio $\left(\dfrac{\text{area of image}}{\text{area of object}}\right)$.

5. (a) Draw axes so that both x and y can take values from -2 to $+8$.
   (b) Draw triangle ABC at A(2, 1), B(7, 1), C(2, 4).
   (c) Find the image of ABC under the transformation represented by the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and plot the image on the graph.
   (d) The transformation is a rotation followed by an enlargement. Calculate the angle of the rotation and the scale factor of the enlargement.

6. (a) Draw axes to that x can take values from 0 to 15 and y can take values from -6 to $+6$.
   (b) Draw triangle PQR at P(2, 1), Q(7, 1), R(2, 4).
   (c) Find the image of PQR under the transformation represented by the matrix $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ and plot the image on the graph.
   (d) The transformation is a rotation followed by an enlargement. Calculate the angle of the rotation and the scale factor of the enlargement.

7. (a) On graph paper, draw the triangle T whose vertices are (2, 2), (6, 2) and (6, 4).
   (b) Draw the image U of T under the transformation whose matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
   (c) Draw the image V of T under the transformation whose matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
   (d) Describe the single transformation which would map U onto V.

8. (a) Find the images of the points (1, 0), (2, 1), (3, −1), (−2, 3) under the transformation with matrix $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$.

(b) Show that the images lie on a straight line, and find its equation.

9. The transformation with matrix $\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$ maps every point in the plane onto a line. Find the equation of the line.

10. Using a scale of 1 cm to one unit in each case draw x- and y-axes, taking values of x from −4 to +6 and values of y from 0 to 12.

(a) Draw and label the quadrilateral OABC with O(0, 0), A(2, 0), B(4, 2), C(0, 2).

(b) Find and draw the image of OABC under the transformation whose matrix is **R**, where $\mathbf{R} = \begin{pmatrix} 2{\cdot}4 & -1{\cdot}8 \\ 1{\cdot}8 & 2{\cdot}4 \end{pmatrix}$.

(c) Calculate, in surd form, the lengths OB and O′B′.

(d) Calculate the angle AOA′.

(e) Given that the transformation **R** consists of a rotation about O followed by an enlargement, state the angle of the rotation and the scale factor of the enlargement.

11. The matrix $\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents a positive rotation of $\theta°$ about the origin. Find the matrix which represents a rotation of:

(a) 90° (b) 180° (c) 30° (d) −90°
(e) 60° (f) 150° (g) 45° (h) 53·1°

Confirm your results for parts (a), (e), (h) by applying the matrix to the quadrilateral O(0, 0), A(0, 2), B(4, 2), C(4, 0).

12. Using the matrix **R** given in question **11**, find the angle of rotation for the following:

(a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0{\cdot}8 & -0{\cdot}6 \\ 0{\cdot}6 & 0{\cdot}8 \end{pmatrix}$

(c) $\begin{pmatrix} 0{\cdot}5 & 0{\cdot}866 \\ -0{\cdot}866 & 0{\cdot}5 \end{pmatrix}$ (d) $\begin{pmatrix} 0{\cdot}6 & 0{\cdot}8 \\ -0{\cdot}8 & 0{\cdot}6 \end{pmatrix}$

Confirm your results by applying each matrix to the quadrilateral O(0, 0), A(0, 2), B(4, 2), C(4, 0).

Exercise 13

1. Draw the rectangle (0, 0), (0, 1), (2, 1), (2, 0) and its image under the following transformations and describe the *single* transformation which each represents:

(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

2. (a) Draw L(1, 1), M(3, 3), N(4, 1) and its image L′M′N′ under the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
   (b) Find and draw the image of L′M′N′ under matrix $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and label it L″M″N″.
   (c) Calculate the matrix product **BA**.
   (d) Find the image of LMN under the matrix **BA**, and compare with the result of performing **A** and then **B**.

3. (a) Draw P(0, 0), Q(2, 2), R(4, 0) and its image P′Q′R′ under matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.
   (b) Find and draw the image of P′Q′R′ under matrix $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and label it P″Q″R″.
   (c) Calculate the matrix product **BA**.
   (d) Find the image of PQR under the matrix **BA**, and compare with the result of performing **A** and then **B**.

4. (a) Draw L(1, 1), M(3, 3), N(4, 1) and its image L′M′N′ under matrix $\mathbf{K} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.
   Find $\mathbf{K}^{-1}$, the inverse of **K**, and now find the image of L′M′N′ under $\mathbf{K}^{-1}$.
   (b) Repeat part (a) with $\mathbf{K} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.
   (c) Repeat part (a) with $\mathbf{K} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

5. The image (x', y') of a point (x, y) under a transformation is given by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix}$
   (a) Find the coordinates of the image of the point (4, 3).
   (b) The image of the point (m, n) is the point (11, 7). Write down two equations involving m and n and hence find the values of m and n.
   (c) The image of the point (h, k) is the point (5, 10). Find the values of h and k.

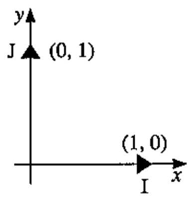
6. Draw A(0, 2), B(2, 2), C(0, 4) and its image under an enlargement, A′(2, 2), B′(6, 2), C′(2, 6).
   (a) What is the centre of enlargement?
   (b) Find the image of ABC under an enlargement, scale factor 2, centre (0, 0).
   (c) Find the translation which maps this image onto A′B′C′.
   (d) What is the matrix **X** and vector **v** which represents an enlargement scale factor 2, centre (−2, 2)?

7. Draw A(0, 1), B(1, 1), C(1, 3) and its image under a reflection A′(4, 1), B′(3, 1), C′(3, 3).
   (a) What is the equation of the mirror line?
   (b) Find the image of ABC under a reflection in the line $x = 0$.
   (c) Find the translation which maps this image onto A′B′C′.
   (d) What is the matrix **X** and vector **v** which represents a reflection in the line $x = 2$?

8. Use the same approach as in questions **6** and **7** to find the matrix **X** and vector **v** which represents each of the following transformations. (Start by drawing an object and its image under the transformation.)
   (a) Enlargement scale factor 2, centre (1, 3)
   (b) Enlargement scale factor 2, centre $(\frac{1}{2}, 1)$
   (c) Reflection in $y = x + 3$
   (d) Rotation 180°, centre $(1\frac{1}{2}, 2\frac{1}{2})$
   (e) Reflection in $y = 1$
   (f) Rotation −90°, centre (2, −2)

Describing a transformation using base vectors

It is possible to describe a transformation in matrix form by considering the effect on the *base vectors* $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.



We will let $\begin{pmatrix}1\\0\end{pmatrix}$ be I and $\begin{pmatrix}0\\1\end{pmatrix}$ be J.

The *columns* of a matrix give us the images of I and J after the transformation.

Example

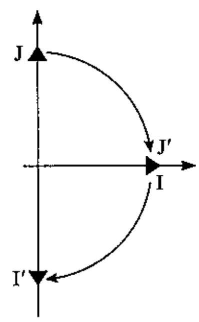
Describe the transformation with matrix $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$.

Column $\begin{pmatrix}0\\-1\end{pmatrix}$ represents I′ (the image of I).

Column $\begin{pmatrix}1\\0\end{pmatrix}$ represents J′ (the image of J)

$$\begin{array}{cc} I' & J' \end{array}$$
$$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$$

Draw I, J, I′ and J′ on a diagram.



Clearly both I and J have been rotated 90° clockwise about the origin. $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ represents a rotation of −90°.

This method can be used to describe a reflection, rotation, enlargement, shear or stretch in which the origin remains fixed.

Exercise 14

In questions **1** to **12**, use base vectors to describe the transformation represented by each matrix.

1. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **2.** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **3.** $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ **4.** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

5. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ **6.** $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ **7.** $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ **8.** $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

In questions **9** to **18**, use base vectors to write down the matrix which represents each of the transformations:

9. Rotation $+90°$ about (0, 0)
10. Reflection in $y = x$
11. Reflection in y-axis
12. Rotation 180° about (0, 0)
13. Enlargement, centre (0, 0), scale factor 3
14. Reflection in $y = -x$
15. Enlargement, centre (0, 0), scale factor -2
16. Reflection in x-axis
17. Rotation $-90°$ about (0, 0)
18. Enlargement, centre (0, 0), scale factor $\frac{1}{2}$

Shear

Figure 1 shows a pack of cards stacked neatly into a pile. Figure 2 shows the same pack after a *shear* has been performed.

Figure 1

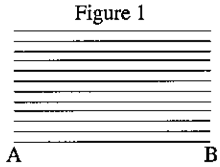
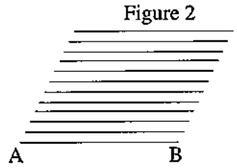


Figure 2

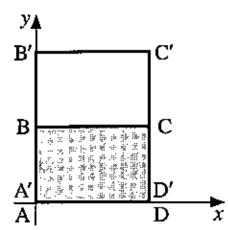


The area is the same after a shear.

Note:
(a) the card AB, at the bottom has not moved (we say the line AB is invariant).
(b) the distance moved by any card depends on its distance from the base card.
The card at the top moves twice as far as the card in the middle.

Stretch

The rectangle ABCD has been *stretched* in the direction of the y-axis so that A′B′ is twice AB.

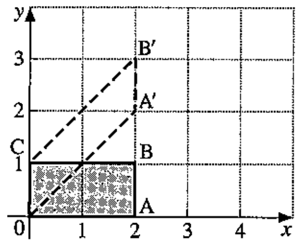


A stretch is fully described if we know:
(a) the direction of the stretch and the invariant line.
(b) the ratio of corresponding lengths.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ represents a stretch parallel to the y-axis, invariant line $y = 0$, where the ratio of corresponding lengths is k.

Exercise 15

1. In the diagram, OABC has been mapped onto OA′B′C by a shear. What is the invariant line of the shear?



2. Draw axes for values of x from -6 to $+9$ and for values of y from -2 to $+5$.

Find the coordinates of the image of each of the following shapes under the shear represented by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Draw each object and image together on a diagram.

(a) (0, 0)(0, 3)(2, 3)(2, 0) (b) (0, 0)(−2, 0)(−2, −2)(0, −2)

(c) (0, 0((2, 2)(3, 0) (d) (1, 1)(1, 3)(3, 3)(3, 1)

What is the invariant line for this shear?

3. Use base vectors to describe the transformation represented by each matrix:

(a) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

In questions **4** to **11**, plot the rectangle ABCD at A(0, 0), B(0, 2), C(3, 2), D(3, 0). Find and draw the image of ABCD under the transformation given and describe the transformation fully.

4. $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ **5.** $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ **6.** $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ **7.** $\begin{pmatrix} 1\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ **9.** $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$ **10.** $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ **11.** $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

12. (a) Find and draw the image of the square (0, 0), (1, 1), (0, 2), (−1, 1) under the transformation represented by the matrix $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$.

(b) Show that the transformation is a shear and find the equation of the invariant line.

13. (a) Find and draw the image of the square (0, 0), (1, 1), (0, 2), (−1, 1) under the shear represented by the matrix $\begin{pmatrix} 0{\cdot}5 & -0{\cdot}5 \\ 0{\cdot}5 & 1{\cdot}5 \end{pmatrix}$.

(b) Find the equation of the invariant line.

14. Find and draw the image of the square (0, 0), (1, 0), (1, 1), (0, 1) under the transformation represented by the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

This transformation is called a two-way stretch.

Revision exercise 9A

1. $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$.

Express as a single matrix:

(a) $2\mathbf{A}$ (b) $\mathbf{A} - \mathbf{B}$ (c) $\frac{1}{2}\mathbf{A}$ (d) $\mathbf{AB}$ (e) $\mathbf{B}^2$

2. Evaluate:

(a) $\begin{pmatrix} -3 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & \frac{1}{3} \\ 1 & \frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}(4 \quad 2)$

(c) $\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} + 2\begin{pmatrix} 3 & 0 \\ -1 & -4 \end{pmatrix}$.

3. $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$, $\mathbf{B} = (1 \quad 5)$, $\mathbf{C} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

(a) Determine $\mathbf{BC}$ and $\mathbf{CB}$.

(b) If $\mathbf{AX} = \begin{pmatrix} 8 & 20 \\ 3 & 7 \end{pmatrix}$, where $\mathbf{X}$ is a (2×2) matrix, determine $\mathbf{X}$.

4. Find the inverse of the matrix $\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$.

5. The determinant of the matrix $\begin{pmatrix} 3 & 2 \\ x & -1 \end{pmatrix}$ is -9.

Find the value of x and write down the inverse of the matrix.

6. $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$; h and k are numbers so that

$\mathbf{A}^2 = h\mathbf{A} + k\mathbf{I}$, where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Find the values of h and k.

7. $\mathbf{M} = \begin{pmatrix} a & 1 \\ 1 & -a \end{pmatrix}$.

(a) Find the values of a if $\mathbf{M}^2 = 17\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Find the values of a if $|\mathbf{M}| = -10$.

8. Find the coordinates of the image of (1, 4) under:

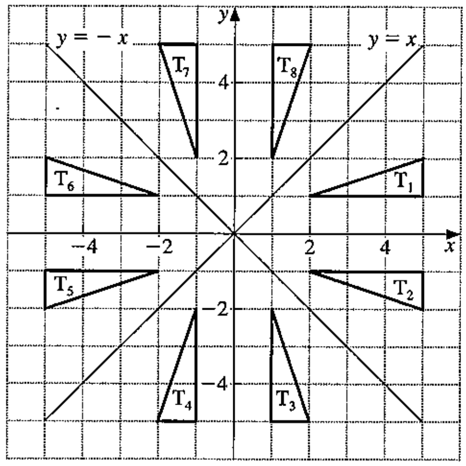
(a) a clockwise rotation of 90° about (0, 0)

(b) a reflection in the line $y = x$

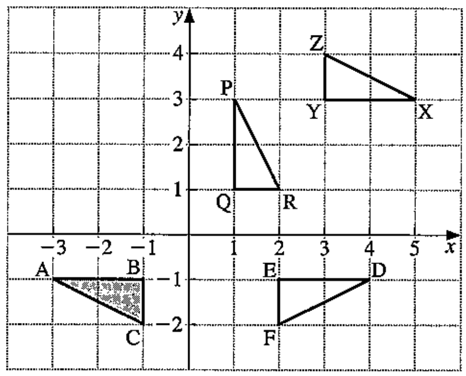
(c) a translation which maps (5, 3) onto (1, 1)

9. Draw x- and y-axes with values from -8 to $+8$. Draw triangle A(1, -1), B(3, -1), C(1, -4). Find the image of ABC under the following enlargements:
(a) scale factor 2, centre (5, -1)
(b) scale factor 2, centre (0, 0)
(c) scale factor $\frac{1}{2}$, centre (1, 3)
(d) scale factor $-\frac{1}{2}$, centre (3, 1)
(e) scale factor -2, centre (0, 0)

10. Using the diagram on the right, describe the transformations for the following:
(a) $T_1 \rightarrow T_6$ (b) $T_4 \rightarrow T_5$
(c) $T_8 \rightarrow T_2$ (d) $T_4 \rightarrow T_1$
(e) $T_8 \rightarrow T_4$ (f) $T_6 \rightarrow T_8$



11. Describe the single transformation which maps:
(a) $\triangle$ABC onto $\triangle$DEF
(b) $\triangle$ABC onto $\triangle$PQR
(c) $\triangle$ABC onto $\triangle$XYZ



12. **M** is a reflection in the line $x + y = 0$.
R is an anticlockwise rotation of 90° about (0, 0).
T is a translation which maps (-1, -1) onto (2, 0).
Find the image of the point (3, 1) under:
(a) **M** (b) **R** (c) **T**
(d) **MR** (e) **RT** (f) **TMR**

13. **A** is a rotation of 180° about (0, 0).
B is a reflection in the line $x = 3$.
C is a translation which maps (3, -1) onto (-2, -1).
Find the image of the point (1, -2) under:
(a) **A** (b) $\mathbf{A}^2$ (c) **BC**
(d) $\mathbf{C}^{-1}$ (e) **ABC** (f) $\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$

14. The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents the transformation **X**.
(a) Find the image of (5, 2) under **X**.
(b) Find the image of (−3, 4) under **X**.
(c) Describe the transformation **X**.

15. Draw x- and y-axes with values from −8 to +8.
Draw triangle A(2, 2), B(6, 2), C(6, 4).
Find the image of ABC under the transformations represented by the matrices:
(a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
(c) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
(e) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Describe each transformation.

16. Using base vectors, describe the transformations represented by the following matrices:
(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

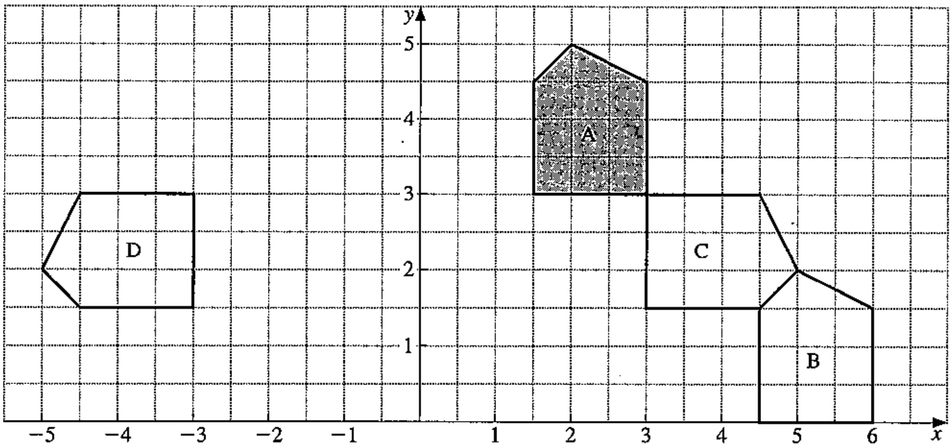
17. Using base vectors, write down the matrices which describe the following transformations:
(a) Rotation 180°, centre (0, 0)
(b) Reflection in the line $y = 0$
(c) Enlargement scale factor 4, centre (0, 0)
(d) Reflection in the line $x = -y$
(e) Clockwise rotation 90°, centre (0, 0)

18. Transformation **N**, which is given by
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix},$$
is composed of two single transformations.
(a) Describe each of the transformations.
(b) Find the image of the point (3, −1) under **N**.
(c) Find the image of the point $(-1, \frac{1}{2})$ under **N**.
(d) Find the point which is mapped by **N** onto the point (7, 4).

19. **A** is the reflection in the line $y = x$.
B is the reflection in the y-axis.
Find the matrix which represents:
(a) **A** (b) **B** (c) **AB** (d) **BA**
Describe the single transformations **AB** and **BA**.

Examination exercise 9B

1.



(a) Describe fully the single transformation which maps the pentagon A onto
(i) B (ii) C (iii) D
(b) Find the matrix of the transformation which maps A onto D.
(c) Describe the single transformation which maps D onto C.
(d) Find the matrix of the transformation which maps D onto C.
(e) Find the equation of the line in which B is reflected onto C.

J 97 4

2. $\mathbf{A} = (1 \quad 2)$, $\mathbf{B} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -2 & 5 \\ -3 & 6 \end{pmatrix}$.

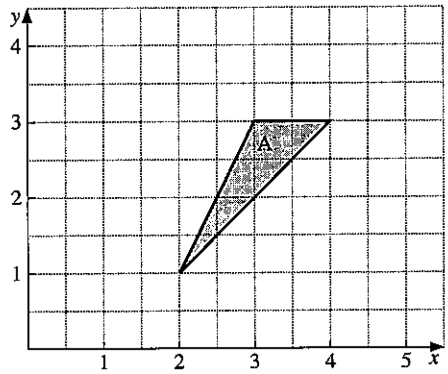
(a) Which one of the following matrix calculations is possible?
(i) $\mathbf{A} + \mathbf{B}$ (ii) $\mathbf{AC}$ (iii) $\mathbf{BC}$
(b) Calculate $\mathbf{AB}$.
(c) Find $\mathbf{C}^{-1}$, the inverse of $\mathbf{C}$.

J 95 2

3. Answer the whole of this question on a sheet of graph paper.
(a) Draw x- and y-axes from -5 to 5 using a scale of 1 cm to represent 1 unit on each axis.
Draw triangle ABC with A(1, 1), B(4, 1) and C(4, 2).
(b) (i) Draw the image of triangle ABC when it is rotated 90° anticlockwise about the origin. Label this image $A_1B_1C_1$.
(ii) Triangle $A_1B_1C_1$ is translated by the vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
Draw and label this image $A_2B_2C_2$.
(iii) Describe fully the single transformation which maps triangle ABC onto triangle $A_2B_2C_2$.
(c) (i) Draw the image of triangle ABC under the transformation represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$. Label this image $A_3B_3C_3$.
(ii) Describe fully the single transformation which maps triangle ABC onto triangle $A_3B_3C_3$.

N 95 4

4. Answer the whole of this question on a sheet of graph paper.



The diagram shows triangle A, with vertices (2, 1), (3, 3) and (4, 3).

(a) Using a scale of 1 cm to represent 1 unit, draw on your graph paper an x-axis for $-6 \leqslant x \leqslant 8$ and a y-axis for $-6 \leqslant y \leqslant 8$. Draw triangle A.

(b) Draw the enlargement of triangle A, centre (0, 0), scale factor 2. Label it B.

(c) Draw the rotation of triangle A, through 90° anticlockwise about (0, 0). Label it C.

(d) Draw the reflection of triangle A in the line $y = -1$. Label it D.

(e) Draw the translation of triangle A by the vector $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$.
Label it E.

(f) (i) A transformation is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
Draw the image of triangle A under this transformation. Label it F.
(ii) Describe fully the single transformation which maps A onto F.

(g) (i) Describe fully the single transformation which maps F onto C.
(ii) Find the matrix for this transformation. J 95 4

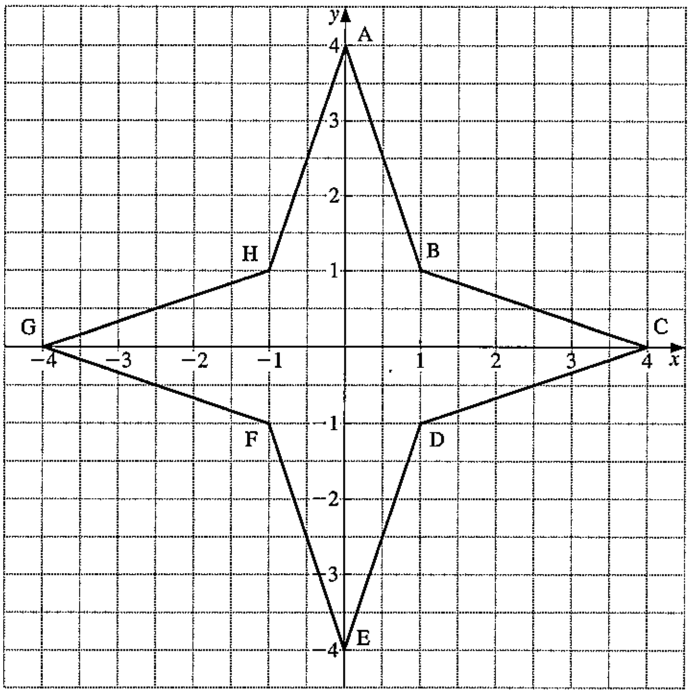
5. Answer the whole of this question on a sheet of graph paper.
Using a scale of 1 cm to represent 1 unit on each axis, draw x- and y-axes from 0 to 16.

(a) On your grid draw triangle T whose vertices are (2, 2), (2, 4) and (6, 4).

(b) Triangle S is the image of triangle T under the transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.
(i) Draw and label triangle S on your diagram.
(ii) Calculate the area of triangle S.
(iii) Describe fully the single transformation represented by the matrix $\mathbf{M}$.

(c) (i) Find $\mathbf{M}^{-1}$, the inverse of the matrix $\mathbf{M}$.
(ii) What is the image of triangle S under the transformation represented by $\mathbf{M}^{-1}$? N 97 4

6.



(a) Describe fully a single transformation which maps:
  (i) both G onto C and H onto B,
  (ii) both G onto D and H onto C,
  (iii) both G onto C and H onto D.

(b) Write down the new positions of the points G and H when they are:
  (i) rotated 90° clockwise about O,
  (ii) reflected in the line $y = x$.

(c) The matrix $\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
  (i) Describe fully the single transformation represented by **M**.
  (ii) Write down the new positions of G and H under **M**.

(d) The matrix $\mathbf{N} = \begin{pmatrix} 4 & -3 \\ -2 & 2 \end{pmatrix}$.
  (i) Find $\mathbf{N}^{-1}$, the inverse of **N**.
  (ii) Write down the positions of G and H after the transformation represented by $\mathbf{NN}^{-1}$. J 98 4

7. (a) Multiply $\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 & -4 \\ 0 & 3 & 6 \end{pmatrix}$.

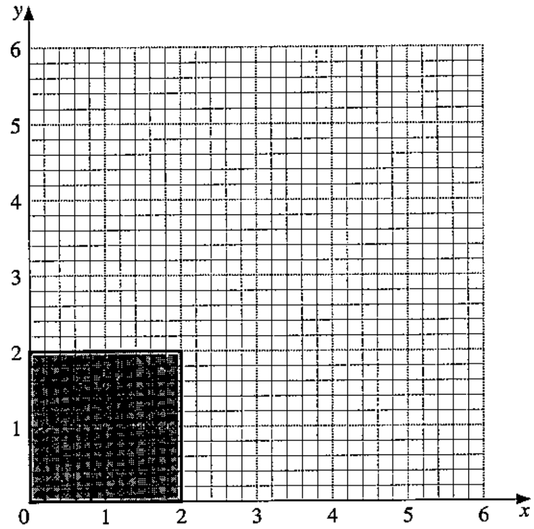
(b) Find the inverse of $\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$.

J 03 2

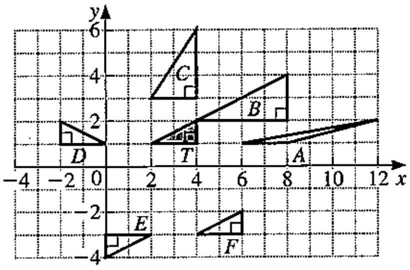
8. (a) Copy the diagram. Draw the shear of the shaded square with the x-axis invariant and the point (0, 2) mapping onto the point (3, 2).

   (b) Make another copy of the diagram in part (a).
   (i) Draw the one-way stretch of the shaded square with the x-axis invariant and the point (0, 2) mapping onto the point (0, 6).
   (ii) Write down the matrix of this stretch.

J 03 2



9. (a)



Use one of the letters A, B, C, D, E or F to answer the following questions.
   (i) Which triangle is T mapped onto by a **translation**? Write down the translation vector.
   (ii) Which triangle is T mapped onto by a **reflection**? Write down the equation of the mirror line.
   (iii) Which triangle is T mapped onto by a **rotation**? Write down the coordinates of the centre of rotation.
   (iv) Which triangle is T mapped onto by a **stretch** with the x-axis invariant? Write down the scale factor of the stretch.
   (v) $\mathbf{M} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$. Which triangle is T mapped onto by $\mathbf{M}$?
   Write down the name of this transformation.

(b) $\mathbf{P} = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} -1 & -2 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, $\mathbf{S} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$.

Only some of the following matrix operations are possible with matrices **P**, **Q**, **R** and **S** above.

PQ, **QP**, **P + Q**, **PR**, **RS**

Write down and calculate each matrix operation that is possible.

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